Mid-term Exam ISI Bangalore September 2017

Differential Topology 100 Points

Notes.
(a) Justify all your steps. You may use any result proved in class unless you have been asked to prove the same.
(b) $\mathbb{R}$ = real numbers.

---

1. [15 points] Let M be an abstract smooth manifold and U, V open subsets in M such that $\overline{U} \cap \overline{V} = \emptyset$. Suppose there is a diffeomorphism $\phi\colon U \to V$. Consider the equivalence relation $\sim$ on M generated by the pairs (x, x), if $x \notin U \cup V$ and $(x, \phi(x))$, if $x \in U$. Let $\overline{M} = M/\sim$. Prove that if $\overline{M}$ is Hausdorff, then it is also a smooth manifold and the canonical map $\pi\colon M \to \overline{M}$ is a local diffeomorphism.

2. [10 points] Let $f(x)$ be a C^∞-function on $\mathbb{R}$. Prove that the locus $y \geq f(x)$ in $\mathbb{R}^2$ is diffeomorphic to the closed upper half plane $y \geq 0$.

3. [10 points] Let X be a manifold, $p \in X$ and V a subspace of $T_p(X)$. Prove that there exists a submanifold Z through p such that $T_p(Z) = V$.

4. [12 points] Let I be an open interval in $\mathbb{R}$. Consider the map $\phi\colon I \to \mathbb{R}^2$ given by $\phi(t) = (\sin(t), \sin(2t))$.
   (i) Verify that ϕ is an immersion.
   (ii) Find an I for which ϕ is not one-one.
   (iii) Find an I for which ϕ is one-one but not an embedding.
   (iv) Find an I for which ϕ is an embedding.

5. [10 points] Prove that a compact manifold does not admit a submersion to $\mathbb{R}^N$.

6. [25 points] Prove that the set X of all 2×2 matrices of rank 1 is a 3 dimensional submanifold of $M(2) = \mathbb{R}^4$. Prove that the set Y of all matrices in $M(2)$ having trace zero is also a 3 dimensional submanifold of $M(2)$ and that X and Y meet transversally.

7. [18 points] Let $f\colon X \to Y$ be a proper smooth map of manifolds with $\dim(X) = \dim(Y)$. Prove that $f^{-1}\{y\}$ is a finite set $\{x_1, \ldots, x_n\}$ for any regular value $y \in Y$. Also show that there exists a neighbourhood U of y in Y such that $f^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_n$ where each V_i is an open neighbourhood of x_i in X and f maps V_i diffeomorphically onto U for each i.